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## On the Functional Definition and Calculation of Global Observables in Nonlinear Spinor Field Quantum Theory

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Nonlinear spinor theory contains unobservable field operators which cannot be identified with free field operators. Therefore for the comparison with experiment a theory of observables for nonlinear spinor fields is required. This theory is developed for global observables by means of a map into functional space, and leads to a functional quantum theory of nonlinear spinor fields.

Nonlinear spinor field theory with noncanonical relativistic Heisenberg quantization is an approach to a unified microscopic description of matter<sup>1,2</sup>. Therefore, in general its spinor field operators  $\Psi_\alpha(x)$  cannot be identified with a special free matter field of conventional Lagrangian coupling field theories, i.e. the spinor field operators are unobservable quantities in a more general way than is assumed usually. As the physical observables have to be defined with respect to the observable quantities of real matter, like asymptotic free particles etc., they can be represented simply only by field operators having a direct physical interpretation. As this is not the case with the nonlinear spinor field, one has to expect therefore a rather complicated connection between these observables and the operators  $\Psi_\alpha(x)$ . From this follows that the methods of construction of observables provided by coupling theories, and even in the wider sense by conventional quantum

field theory, do not suffice to solve this task correctly for nonlinear spinor theory. This is shown in detail in<sup>3</sup> and has been ignored so far. The ignorance of this fact is one of the reasons why dynamical calculations in this theory have not led to a considerable progress in the last decade. Therefore a special theory of observables is required to obtain meaningful physical informations from nonlinear spinor theory. Theoretically, the set of observables is given by local and global quantities. Global observables are defined by the maximal set of simultaneously diagonalizable group generators of the corresponding symmetry groups and by the  $S$ -matrix. Local observables are functionals of the field operators on the position  $x$ . As all observations in microphysics are done really by scattering processes, local observables are not required in principle<sup>4</sup>. Therefore, for a high energy quantum theory the definition and use of global observables is sufficient. In the following we develop therefore a theory only of global observables for the nonlinear spinor field. It will be shown that such a theory is provided by

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the introduction of functional quantum theory<sup>5</sup>, which has been prepared in a series of papers by Stumpf and coworkers<sup>6-28</sup>. To avoid lengthy repetitions in this paper we refer frequently to the preceding papers. Especially we do not repeat the

fundamentals of nonlinear spinor theory in ordinary Hilbert space. They are given in detail in<sup>3,23</sup>. We start immediatly with its functional version. The following definitions are required for the foundation of the functional version of nonlinear spinor theory.

## 1. Definitions

### a) Representation spaces

$$\mathcal{H} := \text{Ordinary Hilbert space of nonlinear spinor theory,} \quad (1.1)$$

$$|a\rangle := \text{State} \in \mathcal{H} \text{ with quantum numbers } a, \quad (1.2)$$

$$|0\rangle := \text{Physical groundstate of the spinor field,} \quad (1.3)$$

$$x := (x_1 x_2 x_3 x_4) \text{ Point of Minkowski space,} \quad (1.4)$$

$$(A_\mu^\nu, d) := \text{Poincare transformation,} \quad (1.5)$$

$$P_\mu, M_{\varrho\sigma} := \text{Infinitesimal generators of the Poincare group,} \quad (1.6)$$

$$L_\alpha^\beta(A) := \text{Hermitean Dirac spinor transformation}^{18}, \quad (1.7)$$

$$P_\mu(x), M_{\varrho\sigma}(x) := \text{Spinor representation of (1.6) in (1.4),} \quad (1.8)$$

$$U(A, d) := \text{Representation of (1.5) in } \mathcal{H}, \quad (1.9)$$

$$\mathbf{P}_\mu, \mathbf{M}_{\varrho\sigma} := \text{Representation of (1.6) in } \mathcal{H}, \quad (1.10)$$

$$|a\rangle' = U(A, d)|a\rangle \text{ Transformation of physical states,} \quad (1.11)$$

$$|a\rangle' = (1 + i\varepsilon^\mu \mathbf{P}_\mu + \tfrac{1}{2} i\omega^{\varrho\sigma} \mathbf{M}_{\varrho\sigma})|a\rangle \text{ General infinitesimal transformation in } \mathcal{H}. \quad (1.12)$$

### b) Operators

$$\Psi_\alpha(x) := \text{Hermitean spinor field operator}^{18,29}, \quad (1.13)$$

$$D_\alpha^\beta(x) := \text{Hermitean Dirac operator for } m = 0^{18,29}, \quad (1.14)$$

$$V_{\alpha\beta\gamma\delta} := \text{Hermitean vertex operator}^{29}, \quad (1.15)$$

$$D_\alpha^\beta(x) \Psi_\beta(x) + V_{\alpha\beta\gamma\delta} : \Psi_\beta(x) \Psi_\gamma(x) \Psi_\delta(x) := 0 \text{ Nonlinear spinor field Eq.}^{18,23,29}, \quad (1.16)$$

$$[\Psi_\alpha(x) \Psi_\beta(x')]_{+/x_0=x_0'} = \varrho_0 \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \text{ Spinor field anticommutator}^{23,29}, \quad (1.17)$$

$$U(A, d) \Psi_\alpha(x') U(A, d)^{-1} = L_\alpha^\beta(A) \Psi_\beta(A^{-1}(x' - d)) \text{ Spinor field transformation law}^{23,29}, \quad (1.18)$$

### c) Global observables

$$\mathbf{\Gamma}^\mu := \frac{1}{2m} \varepsilon^{\mu\nu\varrho\sigma} \mathbf{P}_\nu \mathbf{M}_{\varrho\sigma}, \quad (1.19)$$

$$\mathbf{S}_3 := \text{e}_3\text{-component of total angular momentum in } \mathcal{H}, \quad (1.20)$$

$$\mathbf{P}_\mu, \mathbf{P}^2, \mathbf{\Gamma}^\mu \mathbf{\Gamma}_\mu, \mathbf{S}_3 := \text{Quantum number operators of (1.6) in } \mathcal{H}, \quad (1.21)$$

$$p_\mu, m^2, s, s_3 := \text{Quantum numbers of (1.21),} \quad (1.22)$$

$$|a\rangle = |p, m^2, s, s_3\rangle := \text{Eigenstate of (1.21),} \quad (1.23)$$

$\mathcal{H}^f :=$  Ordinary Hilbert space of nonlinear spinor theory of ingoing or outgoing free particle systems, (1.24)

$|n\rangle :=$  State  $\in \mathcal{H}^f$  with quantum numbers  $n$ , (1.25)

$|a^{(\pm)}(n)\rangle :=$  Scattering states  $\in \mathcal{H}$  with initial resp. final configurations  $n$ , (1.26)

$S_{nm} := \langle a^{(-)}(n) | a^{(+)}(m) \rangle$  Scattering matrix in  $\mathcal{H}$ . (1.27)

#### d) Projections

Time ordered transition matrix element in  $\mathcal{H}$ :

$$\tau_n(x_1 \dots x_n | a) := \langle 0 | T \Psi_{\alpha_1}(x_1) \dots \Psi_{\alpha_n}(x_n) | a \rangle \quad 1 \leq n \leq \infty. \quad (1.28)$$

transformation property of the projections (1.28):

$$\tau_n(x'_1 \dots x'_n | a') = L_{\alpha_1}^{\beta_1}(\Lambda) \dots L_{\alpha_n}^{\beta_n}(\Lambda) \tau_n(\Lambda^{-1}(x'_1 - d) \dots \Lambda^{-1}(x'_n - d) | a) \quad (1.29)$$

The abbreviation

$$g_\alpha(z) f^\alpha(z) := \int g_\alpha(z) f^\alpha(z) d^4z \quad \text{is used in the following}$$

## 2. Functional Map and Isomorphism

After the discussion of the physical and theoretical motivation we turn to the formulation of nonlinear spinor theory by functional quantum theory.

Generally, a quantum theory can be characterized by the property that a definite state space  $\mathcal{A}$  is given, which admits the definition of quantum numbers and a probabilistic interpretation. The quantum number definition itself depends on the existence of corresponding symmetry groups with a maximal set of simultaneously diagonalizable symmetry operators and their eigenvalues. Therefore the state spaces  $\mathcal{A}$  have to be base spaces for representations of these groups with probabilistic interpretation. In conventional quantum theory the explicit representations of the symmetry operators by the dynamical variables e.g. the field operators are known. In noncanonical quantum theories like Heisenbergs approach these operators are not known explicitly in terms of the dynamical variables but they are assumed to exist. So in any case the general property of a quantum theory given above is valid. This property can be considered also to be a definition of a quantum theory. We apply this definition to introduce functional quantum theory.

*Def. 2.1:* A functional quantum theory is given by the assumption of a conventional quantum theory with a state space  $\mathcal{A}$  and by an isomorphic map of  $\mathcal{A}$  into a functional state space  $\mathfrak{A}$ .

*Def. 2.2:* An isomorphic map between  $\mathcal{A}$  and  $\mathfrak{A}$  is given, if any state of  $\mathcal{A}$  can be mapped to a definite state of  $\mathfrak{A}$  and if for all corresponding states the corresponding global observables have the same values.

As we want to deal only with the functional isomorphism of nonlinear spinor theory to a functional quantum theory, we consider only those spaces which are suitable for that problem. To introduce such state spaces, we assume the existence of functional operators  $j_\alpha(x)$  and  $\partial_\alpha(x)$ , where

$$x = \{x_1 x_2 x_3 x_4\}$$

means a point in Minkowski space and  $\alpha$  a spinorial index. Later we discuss the existence of these operators.

We assume the following anticommutation rules

$$\begin{aligned} [j_\alpha(x), j_\beta(x')]_+ &= 0 \\ [j_\alpha(x), \partial_\beta(x')]_+ &= \delta_\alpha^\beta \delta(x - x'), \\ [\partial^\alpha(x), \partial^\beta(x')]_+ &= 0. \end{aligned} \quad (2.1)$$

As the functional space has to be the basic space for an isomorphism to the conventional Hilbert state space, it has to have the same transformation properties as the original space. For an isomorphic map of  $\mathcal{H}$ , therefore, the functional space has to be a representation space of the Poincare group. This leads to the conditions<sup>18</sup>

$$\begin{aligned} V^{-1}(\Lambda, d) j_\alpha(x') V(\Lambda, d) &= L_\alpha^\beta(\Lambda) j_\beta(\Lambda^{-1}(x' - d)), \\ V^{-1}(\Lambda, d) \partial_\alpha(x') V(\Lambda, d) &= L_\alpha^\beta(\Lambda) \partial_\beta(\Lambda^{-1}(x' - d)) \end{aligned} \quad (2.2)$$

for the transformation properties of the functional operators with respect to a Poincare transformation (1.5). The transformation operator  $V$  in functional space will be discussed later. According to this transformation properties, the operators  $j_\alpha(x)$  and  $\partial_\alpha(x)$  have to be Hermitean Dirac spinors with respect to their arguments. In the Hermitean Dirac algebra a notation in analogy to ordinary tensor analysis can be used. With this notation the spinor scalar product  $\psi_\alpha \psi^\alpha$  is an invariant quantity for transformations (1.5), and one may write

$$\partial^\alpha(x) = g^{\alpha\beta} \partial_\beta(x); \quad j^\alpha(x) = g^{\alpha\beta} j_\beta(x) \quad (2.3)$$

where  $g^{\alpha\beta}$  is the metrical fundamental spinor of second rank in spinor space. By this prescription the transformation properties of the quantities (2.3) can be derived also<sup>18</sup>.

The conditions (2.1), (2.2) can be considered to be meaningful only if one succeeds to give an explicit representation of the operators satisfying (2.1), (2.2). A representation satisfying (2.1) merely can be given immediately by the expansion of  $j_\alpha(x)$ ,  $\partial_\alpha(x)$  with a definite testfunction space and the application of a JORDAN-WIGNER algebra<sup>30</sup>. This has been done e.g. by BEREZIN<sup>31</sup>, COESTER<sup>32</sup>. More difficult is the problem to construct a representation which satisfies (2.1) as well as (2.2). Such a represen-

tation has been constructed by STUMPF<sup>18</sup> for spinor functionals and later by RIECKERS<sup>24, 27</sup> for Bose functionals. Only these representations offer the possibility of a functional isomorphism. For the spinor representation can be shown that

$$j_\alpha(x)^+ = \partial_\alpha(x); \quad \partial_\alpha(x)^+ = j_\alpha(x) \quad (2.4)$$

is satisfied by Hermitean conjugation. Further it can be shown that a functional ground state  $|\varphi_0\rangle$  can be constructed with the property

$$\partial_\alpha(x)|\varphi_0\rangle = 0 \quad (2.5)$$

for all  $x$ . From this follows that  $\partial_\alpha(x)$  has to be considered to be a functional destruction operator and  $j_\alpha(x)$  a functional creation operator. These results can be used to construct the functional spaces required. If one applies  $j_\alpha(x)$  repeatedly to  $|\varphi_0\rangle$ , a functional Hilbert space in Fock representation is generated. Defining the power series functionals

$$|D_n(x_1 \dots x_n)\rangle := \frac{1}{n!} j_{\alpha_1}(x_1) \dots j_{\alpha_n}(x_n) |\varphi_0\rangle \\ 0 \leq n < \infty \quad (2.6)$$

the Hermitean conjugate states are given by

$$\langle D_n(x_1 \dots x_n) | := (|D_n(x_1 \dots x_n)\rangle)^+ \quad (2.7)$$

By applying all the rules which have been given, one may derive then the functional scalar product

$$\langle D_n(x_1 \dots x_n) | D_m(x'_1 \dots x'_m) \rangle = \delta_{mn} P \sum_{\lambda_1 \dots \lambda_m} (-1)^P \delta_{\alpha'_1 \lambda_1}^{\alpha_1} \delta(x_1 - x'_1) \dots \delta_{\alpha'_m \lambda_m}^{\alpha_m} \delta(x_m - x'_m) \frac{1}{(n!)^2}. \quad (2.8)$$

Therefore, the set (2.6) defines a functional space equipped with the scalar product (2.8). It can be shown that the set (2.6) is not the only one which can be constructed. This problem will be discussed in Section 5. For the Sections 2, 3, 4 we restrict ourselves to the basic space (2.6).

We are now in the position to define the map of nonlinear spinor theory into functional space.

*Def. 2.3:* The map between the set  $\{|a\rangle\}$  of  $\mathcal{H}$  and the set  $\{|\mathfrak{A}(j, a)\rangle\}$  of the corresponding functional space  $\mathfrak{S}$  of nonlinear spinor theory is defined by

$$|\mathfrak{A}(j, a)\rangle := \sum_{n=0}^{\infty} i^n \tau_n(x_1 \dots x_n | a) |D_n(x_1 \dots x_n)\rangle \quad (2.9)$$

where the set  $\{\tau\}$  is defined by (1.28).

To establish the isomorphism property of this map, several mathematical steps have to be made. For their preparation we prove some statements.

*Stat. 2.1:* For a Poincare transformation (1.5) the representation (1.11) in  $\mathcal{H}$  is mapped into the representation

$$|\mathfrak{A}(j, a')\rangle' = V(A, d) |\mathfrak{A}(j, a)\rangle \quad (2.10)$$

in  $\mathfrak{S}$ .



*Proof:* For the proof we assume the relation  $V|\varphi_0\rangle = |\varphi_0\rangle$  which will be proven later. Then by definition  $|\mathfrak{T}(j, a)\rangle$  is given by (2.9), whereas  $|\mathfrak{T}(j, a')\rangle'$  is defined due to (2.9) by

$$|\mathfrak{T}(j, a')\rangle' := \sum_{n=0}^{\infty} i^n \tau_n(x_1 \dots x_n | a') |D_n(x_1 \dots x_n)\rangle. \quad (2.11)$$

Now we substitute (1.29) into (2.11) and transform the integrals occurring in (2.11) to the new variables  $z_i = A^{-1}(x_i - d)$   $1 \leq i \leq n$ . This gives

$$|\mathfrak{T}(j, a')\rangle' = \sum_{n=0}^{\infty} i^n L_{\alpha_1}^{\beta_1}(A) \dots L_{\alpha_n}^{\beta_n}(A) \tau_n(z_1 \dots z_n | a) |D_n(A z_1 + d, \dots A z_n + d)\rangle. \quad (2.12)$$

Further, from (2.2) follows  $V^{-1}(A, d) j^\alpha(Ax + d) V(A, d) = L_\beta^{\alpha-1}(A) j^\beta(x)$  as  $j^\alpha(x)$  transform contravariant to  $j_\alpha(x)$ . Substitution into (2.12) leads to (2.10), q.e.d.

*Stat. 2.2:* The infinitesimal generator  $\delta V$  of  $V$  for a general infinitesimal transformation (1.12) reads

$$(1 + \delta V) := (1 + i \varepsilon^\mu \mathfrak{P}_\mu + \frac{1}{2} i \omega^{\varrho\sigma} \mathfrak{M}_{\varrho\sigma}) \quad (2.13)$$

with the operators

$$\mathfrak{P}_k := j_\alpha(x) P_k(x) \partial^\alpha(x); \quad \mathfrak{M}_{kl} := j_\alpha(x) M_{kl}(x) \partial^\alpha(x) \quad (2.14)$$

where  $P_k$  and  $M_{kl}$  are the generators in ordinary spinor space.

*Proof:* We consider the identity

$$\begin{aligned} \tau_n(x_1 \dots x_n) |D_n(x_1 \dots x_n)\rangle &= \tau_n(A^{-1}(x'_1 - d) \dots A^{-1}(x'_n - d)) |D_n(A^{-1}(x'_1 - d) \dots A^{-1}(x'_n - d))\rangle \\ &= \tau_n(A^{-1}(x'_1 - d) \dots A^{-1}(x'_n - d)) L_{\beta_1}^{\alpha_1}(A) \dots L_{\beta_n}^{\alpha_n}(A) V^{-1}(A, d) |D_n(x'_1 \dots x'_n)\rangle \end{aligned} \quad (2.15)$$

valid for arbitrary antisymmetric functions  $\tau_n$  with the transformation property (1.29). For the general infinitesimal transformation (1.12) we obtain from (1.29) with  $(A, d) := (\omega^{\varrho\sigma} M_{\varrho\sigma}, \varepsilon^\mu P_\mu)$

$$L_{\beta_1}^{\alpha_1}(A) \dots L_{\beta_n}^{\alpha_n}(A) \tau_n(A^{-1}(x'_1 - d) \dots A^{-1}(x'_n - d)) = \left(1 + \sum_{k=1}^n \left[ i \varepsilon^\mu P_\mu(x'_k) + \frac{i}{2} \omega^{\varrho\sigma} M_{\varrho\sigma}(x'_k) \right] \right) \tau_n(x'_1 \dots x'_n) \quad (2.16)$$

where we omitted for brevity the spin indices at the right side and no summation has to be performed in  $x'$ . Writing  $V^{-1} = 1 - \delta V$  and expanding  $\delta V$  in a power series of  $\varepsilon^\mu$  and  $\omega^{\varrho\sigma}$  by substitution into (2.15) together with (2.16) follows (2.13) in first order terms, q.e.d.

From stat. 2.2 immediately follows  $V|\varphi_0\rangle = |\varphi_0\rangle$ , as for the infinitesimal  $\delta V$  this relation is true and any finite  $V$  can be constructed by repeated application of  $\delta V$ . Therefore, the assumption of stat. 2.1 is proven.

*Stat. 2.3:* If  $|a\rangle \in \mathcal{H}$  is an eigenstate of (1.21) then  $|\mathfrak{T}(j, a)\rangle$  is an eigenstate of

$$\mathfrak{P}_k |\mathfrak{T}(j, a)\rangle = p_k |\mathfrak{T}(j, a)\rangle; \quad \mathfrak{P}^2 |\mathfrak{T}(j, a)\rangle = m^2 |\mathfrak{T}(j, a)\rangle, \quad (2.17)$$

$$\mathfrak{G}_\mu \mathfrak{G}^\mu |\mathfrak{T}(j, a)\rangle = s(s+1) |\mathfrak{T}(j, a)\rangle; \quad \mathfrak{S}_3 |\mathfrak{T}(j, a)\rangle = s_3 |\mathfrak{T}(j, a)\rangle \quad (2.18)$$

with

$$\mathfrak{G}^\mu := (1/2m) \varepsilon^{\mu\varrho\sigma} \mathfrak{P}_\varrho \mathfrak{M}_{\sigma\mu} \text{ and corresponding definition of } \mathfrak{S}_3. \quad (2.19)$$

*Proof:* Is given in <sup>21</sup>.

The symmetry conditions discussed so far define the set of possible quantum numbers. The selection of those quantum numbers out of this set which occur really in nature (or least in the theory) is effected by the dynamics of the system, i.e. the field Eq. (1.16). Also this equation has to be mapped in functional space to formulate the dynamics in this space.

*Stat. 2.4:* The functional map of the field Eq. (1.16) is given by the functional equation

$$[D_{\alpha\beta}(x) \partial_{\beta}(x) + V_{\alpha\beta\gamma\delta} [\partial_{\beta}(x) \partial_{\gamma}(x) \partial_{\delta}(x) + 3 F_{\beta\gamma}(0) \partial_{\delta}(x)] - i \rho_0 j_{\alpha}(x)] |\mathfrak{T}(j)\rangle = 0 \quad (2.20)$$

which has to be defined in the functional space of the set (2.6).

*Proof:* Is given in<sup>5,18</sup>.

Now, for this map it is possible to define the noncanonical Heisenberg quantization without ambiguity. To do this, we apply first the causal Green function  $D^{-1}(x)$  to (2.20). By spectral decomposition it can be shown that the inhomogenous term has to be zero if the corresponding state functional  $|\mathfrak{T}(j)\rangle$  belongs to a representation (2.17) with  $m \neq 0$ . Therefore in this case the equation

$$[\partial_{\alpha}(x) + G_{\alpha}^{\alpha}(x - x') \{V_{\alpha\beta\gamma\delta} [\partial_{\beta}(x') \partial_{\gamma}(x') \partial_{\delta}(x') + 3 F_{\beta\gamma}(0) \partial_{\delta}(x')] - i \rho_0 j_{\alpha}(x')\}] |\mathfrak{T}(j)\rangle = 0 \quad (2.21)$$

results. Then we introduce the normal transform

$$|\mathfrak{T}(j)\rangle =: \exp \{-j_{\alpha}(x) F^{\alpha\beta}(xy) j_{\beta}(y)\} |\Phi(j)\rangle \quad (2.22)$$

where  $F^{\alpha\beta}(xy)$  is the corresponding two point function of the theory. By substitution of (2.22) into (2.21) the equation

$$[\partial_{\alpha}(x) + F_{\alpha}^{1\alpha}(xx') j_{\alpha}(x') + G_{\alpha}^{\alpha}(x - x') V_{\alpha\beta\gamma\delta} d_{\beta}(x') d_{\gamma}(x') d_{\delta}(x')] |\Phi(j)\rangle = 0 \quad (2.23)$$

follows with

$$d_{\alpha}(x) := F_{\alpha}^{1\alpha}(xz) j_{\alpha}(z) + \partial_{\alpha}(x) \quad \text{and} \quad F^{1\alpha}(xx') := F_{\alpha}^{\alpha}(xx') - i \rho_0 G_{\alpha}^{\alpha}(x - x'). \quad (2.24), (2.25)$$

One derives easily also from (2.17), (2.18) by substitution of (2.22)

$$\mathfrak{P}_k |\Phi(j, a)\rangle = p_k |\Phi(j, a)\rangle; \quad \mathfrak{P}^2 |\Phi(j, a)\rangle = m^2 |\Phi(j, a)\rangle, \quad (2.26)$$

$$\mathfrak{G}_{\mu} \mathfrak{G}^{\mu} |\Phi(j, a)\rangle = s(s+1) |\Phi(j, a)\rangle; \quad \mathfrak{S}_3 |\Phi(j, a)\rangle = s_3 |\Phi(j, a)\rangle. \quad (2.27)$$

Instead of  $|\mathfrak{T}(j)\rangle$  one may use, therefore, equivalently  $|\Phi(j)\rangle$  with the corresponding conditions resulting from the functional map. It is remarkable that in this version the total information about the commutation rules (1.17) is expressed by the Poincaré-invariant functions  $F$  and  $F^1$ . Therefore a change of the commutators (1.17) leads to a change of  $F$  and  $F^1$ . But the reverse is also true. For canonical quantization  $F$  and  $F^1$  have to be singular distributions on the lightcone in  $z = x' - x$ <sup>33,34</sup>. As is known that these singularities are tightly connected with the field theoretic divergencies, it is plausible to start with a regularization of  $F$  and  $F^1$ . This regularization then causes a change in the quantization (1.17). But this is of no special interest. The importance of the transformation (2.22) is due to the fact that by transition from canonical to noncanonical quantization no structural change occurs in the functional map, while on the contrary in the original Hilbert space the entire mathematical apparatus of conventional, i.e. canonical quantum theory breaks down. Therefore a noncanonical quantization may be defined without ambiguity in the

functional  $|\Phi\rangle$ -map. As this cannot be performed in ordinary Hilbert space, this is the first hint about the importance of functional quantum theory. Of course the noncanonical change of  $F$  resp.  $F^1$  cannot be done arbitrarily. There are certain restrictions, which are called selfconsistency conditions. For the treatment of these conditions we refer to the literature<sup>10,35</sup> as we are not interested in them for this investigation. Concerning the definition of the observables we shall refer always to the  $|\mathfrak{T}(j)\rangle$ -functionals. But this is no restriction. As it is assumed that  $F$  resp.  $F^1$  are given and well defined (just by the introduction of noncanonical quantization), to each  $|\Phi(j)\rangle$  also the corresponding  $|\mathfrak{T}(j)\rangle$  may be constructed. Therefore one may use the  $|\mathfrak{T}(j)\rangle$  functionals equivalently.

### 3. Global Functional Observables

In this section we establish the isomorphism property of the map (2.9) to the spinorfield quantum theory of Section 1. There the global observables are given by (1.21) and by (1.27). To reproduce

these observables by the functional map (2.9), we have to introduce a functional scalar product for the physical state functionals  $|\mathfrak{T}(j)\rangle \in \mathfrak{F}$ . Using such a scalar product, one has to observe that even in ordinary quantum theory this product is an axiomatic definition. Therefore it is convenient to introduce the functional scalar product in  $\mathfrak{F}$  also by axiomatic definition, which has to be justified later by the proof of the isomorphism property. It will turn out that this is the only reasonable procedure. As a proper axiomatic definition of the scalar product surely cannot be arbitrary, we look out for hints by studying the scalar product in ordinary Hilbert space. As one observes easily in this space some general conditions are satisfied by the product. We generalize these conditions to axiomatic statements which have to be satisfied commonly by every isomorphism to other spaces. As the scalar product is defined by the definition of the metrical fundamental tensor in the corresponding state space, these axioms concern the definition of that tensor. They read:

*Axioms:* The metrical fundamental tensor in a given physical quantum state space has

$$\begin{aligned} \tilde{\tau}_n(q_1 \dots q_n | a) = & P \sum_{\lambda_1 \dots \lambda_n} \sum_{\mu_1 \dots \mu_{n-1}} (-1)^P M(\mu_1 \dots \mu_n | a) \\ & \times \delta \left( \sum_{s=1}^n q_{\lambda_s} - p_{\mu_n} \right) \prod_{r=1}^{n-1} \delta \left( \sum_{\beta=1}^r q_{\lambda_\beta} - p_{\mu_r} + i\varepsilon \right)^{-1} \end{aligned} \quad (3.1)$$

where  $p_\mu \equiv (p_\mu, p_\mu^4)$  is the fourmomentum of the intermediate state  $|\mu\rangle$  and  $M$  are the corresponding matrixelements. As for the derivation of (3.1) only the transformation properties (1.29) are used, this formula is valid even if unphysical intermediate states occur, i.e. if  $p_{\mu_\alpha}$  is spacelike and the corresponding mass  $\mu_\alpha$  imaginary. On the other hand, by the summation used in (3.1), it is indicated that the spectral decomposition is confined to those cases, where only discrete intermediate mass spectra are present. If continuous mass-spectra occur, the corresponding spectral function  $\varrho(\mu^2)$  has to be substituted by a discrete mass distribution reproducing all moments of  $\varrho(m^2)$ . Therefore this is no severe restriction. It reflects only the difficulty of a probabilistic interpretation of continuous mass spectra, which is circumvented by their reduction to a discrete representation.

Having explained the decomposition formula (3.1) sufficiently, we may turn to the scalar product de-

- a) to be forminvariant with respect to symmetry operations of the corresponding invariance groups,
- b) to be universal, i.e. applicable to all quantum models with the same symmetry groups,
- c) to orthonormalize all irreducible base representations of the symmetry groups in this space.

To apply these axioms to the functional map (2.9), one has to observe that this map works also for other relativistic invariant spinor theories, like free Fermifields or spinor coupling theories. In any case, it is assumed that the state space of the original theory is a representation space of the Poincare group. Then, by statement 2.1, the functional space of physical state functionals is a representation space of the Poincare group, too. Therefore the condition of the application of the axioms a), b) c) is satisfied. We consider first the third axiom c). To draw conclusions from it we perform a general spectral decomposition of the generating functions (1.28). For convenience we evaluate this decomposition in Fourier space. Denoting the Fouriertransforms of (1.28) by  $\tilde{\tau}_n$   $1 \leq n < \infty$ , the following representation can be derived<sup>26</sup>, App. I

finition. Among the intermediate states  $\{|\mu\rangle\}$  also the groundstate  $|\mu\rangle = |0\rangle$  with  $p_\mu = 0$  may occur. Assuming the metrical fundamental tensor to be known already, it can be shown by direct calculation that a necessary condition for the finiteness of the scalar products is the removal of the intermediate states  $|\mu\rangle = |0\rangle$ . This can be done constructively by the evaluation of a given  $|\mathfrak{T}(j)\rangle$  in the standard form (3.1) and omission of all terms with

$$|\mu_i\rangle = |0\rangle \quad i = 1 \dots n.$$

The resulting  $\tilde{\tau}$  function we denote by  $\tilde{\tau}_n^r(q_1 \dots q_n)$  where the subscript  $r$  indicates this procedure. This function arises from (3.1) if one substitutes for  $\sum_{\mu_1 \dots \mu_{n-1}}$  the reduced summation  $\sum_{\mu_1 \dots \mu_{n-1}}^r$ . The corresponding reduced state functional then is given by

$$|\mathfrak{T}(\tilde{j}, a)\rangle^r := \sum_{n=1}^{\infty} i^n \tilde{\tau}_n^r(q_{1n}^r \dots q_n) |D_n(q_1 \dots q_n)\rangle \quad (3.2)$$

where  $|D_n(q_1 \dots q_n)\rangle$  is the functional Fouriertransform of (2.6), and  $\tilde{j} := f(j)$  is the functional Fourier-Transform of  $j(x)$ . All properties of the power series functionals (2.6) in ordinary space are reproduced in Fourier space. After this preparation the following can be proven:

*Stat. 3.1:* If in the spectral decomposition of two physical state functionals  $|\mathfrak{T}(j, a)\rangle$  and  $|\mathfrak{T}(j, b)\rangle$  only discrete intermediate mass states occur, their functional scalar product satisfying axioms a) and b) has to be defined by

$$(\mathfrak{T}(j, a) | \mathfrak{T}(j, b)) := \lim_{\varepsilon \rightarrow 0} {}^r \langle \mathfrak{T}(\tilde{j}, a) | \mathfrak{B} | \mathfrak{T}(\tilde{j}, b) \rangle^r \quad (3.3)$$

with

$$\mathfrak{B} := \sum_{m=1}^{\infty} \varepsilon^m |D_m(q_1 \dots q_m)\rangle \delta_{\beta_1}^{\gamma_1} \delta(q_1 - p_1) \dots \delta_{\beta_m}^{\gamma_m} \delta(q_m - p_m) \langle D_m(p_1 \dots p_m) | \quad (3.4)$$

where  $\varepsilon$  has to be identified with that of (3.1).

*Proof:* We consider first the most general expression for a metrical fundamental tensor in functional space, which is given by  $\mathfrak{B} = \sum_{n,m} |D_n\rangle w_{nm} \langle D_m|$  with arbitrary but symmetric

$$w_{nm} := w_{nm}(q_1 \dots q_n, p_1 \dots p_m)$$

Due to axiom b)  $\mathfrak{B}$  has to be applicable to free relativistic Fermifield functionals<sup>18</sup>. A direct calculation shows that then  $\mathfrak{B}$  has to have the form (3.4). Applying a Poincare transformation by the rules given in Section 2 the transformed  $\mathfrak{B}$  has to be  $\mathfrak{B}' = V \mathfrak{B} V^{-1}$  from which easily  $\mathfrak{B}' \equiv \mathfrak{B}$  can be derived. Therefore  $\mathfrak{B}$  is forminvariant under the

corresponding symmetry group and satisfies axiom a) too, q.e.d.

With the scalar product (3.3) several statements can be proven.

*Stat. 3.2:* Functional eigenstates  $|\mathfrak{T}(j, a)\rangle$  and  $|\mathfrak{T}(j, b)\rangle$  of (2.17) and (2.18) are orthogonal for  $|a\rangle \neq |b\rangle$ .

*Proof:* For simplicity we assume the set (2.17), (2.18) to be complete, i.e. we omit other quantum numbers which do not result from the Poincare-group. Then no degeneracy may occur. Therefore if  $|a\rangle \neq |b\rangle$ , among the set of quantum numbers  $a := (a_1 \dots a_i)$  for the state  $|a\rangle$  and  $b := (b_1 \dots b_i)$  for the state  $|b\rangle$  at least one pair  $(a_\alpha, b_\alpha)$  is different. As the quantum numbers are defined by the diagonalization of the generators, we denote the corresponding generator by  $J_\alpha$ . In order to work with the reduced functionals  $|\mathfrak{T}(j, a)\rangle^r$ , it can be shown<sup>29</sup> that

$$|\mathfrak{T}(j, a)\rangle = \mathfrak{T}(j, 0) | \mathfrak{T}(j, a) \rangle^r \quad (3.5)$$

is valid, where  $\mathfrak{T}(j, 0)$  is the vacuum functional. As we did derive  $|\mathfrak{T}(j, a)\rangle^r$  by direct construction, we do not need  $\mathfrak{T}(j, 0)$  explicitly, but use (3.5) only formally. As for  $\mathfrak{T}(j, 0)$  all quantum numbers vanish, it commutes with any  $J_\alpha$ . Therefore from

$$J_\alpha |\mathfrak{T}(j, a)\rangle = a_\alpha |\mathfrak{T}(j, a)\rangle$$

follows by (3.5) for the reduced functional the condition

$$J_\alpha |\mathfrak{T}(j, a)\rangle^r = a_\alpha |\mathfrak{T}(j, a)\rangle^r \quad (3.7)$$

and an equivalent one for  $|\mathfrak{T}(j, b)\rangle^r$ . Then we may apply the scalar product definition to these eigenvalue conditions and obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} {}^r \langle \mathfrak{T}(\tilde{j}, b) | \mathfrak{B} J_\alpha | \mathfrak{T}(\tilde{j}, a) \rangle^r &= a_\alpha \lim_{\varepsilon \rightarrow 0} {}^r \langle \mathfrak{T}(\tilde{j}, b) | \mathfrak{B} | \mathfrak{T}(\tilde{j}, a) \rangle^r, \\ \lim_{\varepsilon \rightarrow 0} {}^r \langle \mathfrak{T}(\tilde{j}, a) | \mathfrak{B} J_\alpha | \mathfrak{T}(\tilde{j}, b) \rangle^r &= b_\alpha \lim_{\varepsilon \rightarrow 0} {}^r \langle \mathfrak{T}(\tilde{j}, a) | \mathfrak{B} | \mathfrak{T}(\tilde{j}, b) \rangle^r. \end{aligned} \quad (3.8)$$

By direct calculation follows

$${}^r \langle \mathfrak{T}(\tilde{j}, a) | \mathfrak{B} | \mathfrak{T}(\tilde{j}, b) \rangle^r = {}^r \langle \mathfrak{T}(\tilde{j}, b) | \mathfrak{B} | \mathfrak{T}(\tilde{j}, a) \rangle^r \times \quad (3.9)$$

where  $\times$  means complex conjugation. Therefore, from (3.8) and (3.9) one obtains immediatly

$$(\mathfrak{T}(j, b) | J_\alpha \mathfrak{T}(j, a)) - (\mathfrak{T}(j, a) | J_\alpha \mathfrak{T}(j, b)) \times = (a_\alpha - b_\alpha) (\mathfrak{T}(j, b) | \mathfrak{T}(j, a)). \quad (3.10)$$

As  $J_\alpha$  leads to an observable quantum number, we may assume the corresponding  $J_\alpha(x)$  in configuration space to be a Hermitean operator. Then, by

direct calculations follows

$$(\mathfrak{T}(j, b) | J_\alpha \mathfrak{T}(j, a)) = (\mathfrak{T}(j, a) | J_\alpha \mathfrak{T}(j, b)) \times \quad (3.11)$$



and therefore from (3.10)

$$(\mathfrak{T}(j, b) | \mathfrak{T}(j, a)) = 0 \quad (3.12)$$

for  $a \neq b$  q.e.d.

Choosing a suitable normalization one therefore has with respect to (3.3) a unitary map between the eigenstates  $|\mathfrak{T}(j, a)\rangle \in \mathfrak{H}$  and  $|a\rangle \in \mathcal{H}$ . This map can be extended to scattering states also. We prove the following:

*Stat. 3.3:* If  $|a\rangle$  is a base vector of an irreducible representation of the Poincare-group in  $\mathcal{H}$ , then  $|\mathfrak{T}(j, a)\rangle$  is a base vector of the corresponding irreducible representation in  $\mathfrak{H}$ .

*Proof:* The irreducibility of a representation does depend only on their quantum numbers. Therefore, if  $|a\rangle$  belongs to an irreducible representation with a maximal set of quantum numbers (1.22) due to Stat. 2.3, also  $|\mathfrak{T}(j, a)\rangle$  exhibits this quantum numbers. But then, due to the characterization of the irreducibility, also  $|\mathfrak{T}(j, a)\rangle$  has to be a base vector of an irreducible representation q.e.d.

*Stat. 3.4:* If  $|a\rangle$  and  $|b\rangle$  are orthogonal base vectors of irreducible representations in  $\mathcal{H}$ , the same is true for  $|\mathfrak{T}(j, a)\rangle$  and  $|\mathfrak{T}(j, b)\rangle$  in  $\mathfrak{H}$  with respect to (3.3).

*Proof:* As in irreducible representations no degeneracy occurs, any state  $|a\rangle$  of such a representation has its specific quantum numbers, different from the quantum numbers of all other states of irreducible representations. Therefore the Stat. 3.2 can be applied to give this statement, q.e.d.

*Stat. 3.5:* Between scattering states  $|a^{(\pm)}(n)\rangle \in \mathcal{H}$  and the corresponding scattering states

$$|\mathfrak{T}^{(\pm)}(j, n)\rangle \in \mathfrak{H}$$

a unitary mapping is provided by (3.3).

*Proof:* Any physical state has to be a base state of a representation of the corresponding symmetry groups. Contrary to the eigenstates, the scattering states generally do not generate irreducible but only reducible representations. As the entire representation space is composed of the direct sum of all base vectors of irreducible representations, it follows that any scattering state can be decomposed into irreducible parts

$$|a^{(\pm)}(n)\rangle = \sum_{\alpha} C_{\alpha}^{(\pm)}(n) |\alpha\rangle^{\text{irr}} \quad (3.13)$$

where  $|\alpha\rangle^{\text{irr}}$  has to be a base vector of an irreducible representation but in generally no eigenstate of the dynamical equation! By substitution of (3.13) in (2.9) and by Stat. 3.3, then follows immediately

$$|\mathfrak{T}^{(\pm)}(j, n)\rangle = \sum_{\alpha} C_{\alpha}^{(\pm)}(n) |\mathfrak{T}(j, \alpha)\rangle^{\text{irr}}. \quad (3.14)$$

Applying Stat. 3.4, from (3.13) and (3.14) follows the relation

$${}^{\text{irr}}\langle \beta | a^{(\pm)}(n) \rangle = C_{\beta}^{(\pm)}(n) = {}^{\text{irr}}(\mathfrak{T}(j, \beta) | \mathfrak{T}^{(\pm)}(j, n)). \quad (3.15)$$

Therefore any scattering state can be decomposed uniquely into irreducible parts, which are mapped unitary. From this follows the statement immediately, q.e.d.

Now we turn to the global observables. We define the  $S$ -matrix in functional space by

*Def. 3.1:* For two scattering states  $|\mathfrak{T}^{(+)}(j, n)\rangle$  and  $|\mathfrak{T}^{(-)}(j, m)\rangle$  in  $\mathfrak{H}$  the  $S$ -matrix is given by the scalarproduct

$$S_{mn} := (\mathfrak{T}^{(-)}(j, m) | \mathfrak{T}^{(+)}(j, n)). \quad (3.16)$$

We have to show the physical meaning of this definition. From the statements proven in the foregoing it follows immediately the

*Stat. 3.6:* The definition of the  $S$ -matrix (1.27) in  $\mathcal{H}$  is invariant on the map (2.9) of  $\mathcal{H}$  into  $\mathfrak{H}$ .

*Proof:* We substitute the expansion (3.13) into (1.27). Then one obtains, due to the orthonormality of irreducible base vectors in  $\mathcal{H}$

$$S_{mn} = \sum_{\alpha} C_{\alpha}^{(-)}(m) C_{\alpha}^{(+)}(n). \quad (3.17)$$

The same result is obtained by substitution of (3.14) into (3.16). Due to Stat. 3.4 also in  $\mathfrak{H}$  the irreducible base vectors are orthonormalized and (3.17) results, q.e.d.

Summarizing the results of the preceding statements the following can be concluded:

*Stat. 3.7:* By the map (2.9) and the scalarproduct (3.3) an isomorphic map between the representation of the spinor theory in  $\mathcal{H}$  and  $\mathfrak{H}$  is established.

*Proof:* The global observables are defined by the maximal set of quantum numbers and by  $S_{ab}$ . Applying the preceding statements in both spaces, one obtains for corresponding states the same global observables. By definition 2.2 this establishes the isomorphism, q.e.d.

#### 4. Functional Calculation Methods

In the preceding section the isomorphism property of the map (2.9) together with the scalar-product (3.3) was proven. But this isomorphism is as long without any practical value, as one cannot calculate the maps of the physical states in functional space. Therefore we have to discuss calculational procedures for state functionals. To develop these procedures, functional relativistic cluster representations have to be used and the problem of imposing appropriate boundary conditions<sup>3</sup> has to be solved. As will be seen in the following, these procedures resemble strongly those used in the non-relativistic cluster theory by EKSTEIN<sup>36</sup> and WILDERMUTH<sup>37</sup>. Therefore it is convenient to use a functional equation which is the relativistic analogy to the time independent Schrödinger equation. We perform all operations in the Normal-representation, as by this representation in nonlinear spinor theory the noncanonical quantization is defined. But for the calculation of the corresponding observables we return always to the time ordered representation (2.9). Applying to (2.23) the operator  $j^\kappa(x) P_h(x)$  one obtains

$$j^\kappa(x) P_h(x) \partial_\kappa(x) |\Phi(j)\rangle = -\mathfrak{D}_h(j, d) |\Phi(j)\rangle \quad (4.1)$$

with

$$\begin{aligned} \mathfrak{D}_h(j, d) := & j^\kappa(x) P_h(x) [F^1(xz) j^\alpha(z) \\ & + G(xz) V^{\alpha\beta\gamma\delta} d_\beta(z) d_\gamma(z) d_\delta(z)] \end{aligned} \quad (4.2)$$

In the following we assume  $|\Phi(j)\rangle$  always to be an eigenstate with respect to the total mass of the system in consideration. This assumption is consistent also with scattering processes. Then one may derive the eigenvalue equation required. As  $\mathfrak{P}^h$  commutes with  $\mathfrak{D}_h$ , one may apply it to (4.1). By use of (2.26), then from (4.1) results the equation

$$[m^2 - \mathfrak{D}^h(j, d) \mathfrak{D}_h(j, d)] |\Phi(j)\rangle = 0. \quad (4.3)$$

Although this equation is only a necessary and not a sufficient dynamical condition, we use it as a simple representation of the dynamical law. It is meaningful as long as states with  $m \neq 0$  are treated. The case of mass-zero-particles or systems we discuss later.

##### a) Bound state calculations

Due to baryon conservation which we do not discuss here explicitly for any one-particle-bound state,

there exists a smallest  $\varrho$  with  $\varphi_\varrho \neq 0$ , whereas all  $\varphi_n$  with  $n < \varrho$  disappear. Therefore in this sector the normal functional may be written

$$|\Phi(j)\rangle = \sum_{n=\varrho}^{\infty} \varphi_n(x_1 \dots x_n) |D_n(x_1 \dots x_n)\rangle. \quad (4.4)$$

Now we define the projection operators

$$\begin{aligned} P_k := & |D_k(z_1 \dots z_k)\rangle \delta_{\beta_1}^{\gamma_1} \delta(z_1 - y_1) \\ & \dots \delta_{\beta_k}^{\gamma_k} \delta(z_k - y_k) \langle D_k(y_1 \dots y_k)| \end{aligned} \quad (4.5)$$

where (4.5) has not to be confused with (1.6) or (1.8). Defining further

$$|\Phi_\varrho(j)\rangle := P_\varrho |\Phi(j)\rangle \quad (4.6)$$

the state functional (4.4) can be written also

$$|\Phi(j)\rangle = |\Phi_\varrho(j)\rangle + |\Phi_r(j)\rangle \quad (4.7)$$

where  $|\Phi_r\rangle$  is the complement of  $|\Phi_\varrho\rangle$  with respect to  $|\Phi\rangle$ . By substitution of (4.7) into (4.3) a set of coupled functional equation for  $|\Phi_\varrho\rangle$  and  $|\Phi_r\rangle$  can be derived. Eliminating  $|\Phi_r\rangle$  from this set, an equation for  $|\Phi_\varrho\rangle$  only follows<sup>23</sup>. This equation reads

$$\begin{aligned} [(m^2 - P_\varrho \mathfrak{D}_h \mathfrak{D}^h P_\varrho) - P_\varrho (m^2 - \mathfrak{D}_h \mathfrak{D}^h) \Pi_\varrho \\ \cdot (m^2 - \Pi_\varrho \mathfrak{D}_h \mathfrak{D}^h \Pi_\varrho)^{-1} \Pi_\varrho (m^2 - \mathfrak{D}_h \mathfrak{D}^h) P_\varrho] |\Phi_\varrho\rangle = 0 \end{aligned} \quad (4.8)$$

and has to be solved. Additionally for bound states, i.e. for irreducible representations, the conditions (2.26), (2.27) have to be satisfied. It has been shown by SCHULER and STUMPF<sup>14</sup> that these conditions are just satisfied if the maximal set of quantum number operators commutes with the operator of the dynamical equation and if  $|\Phi_\varrho\rangle$  itself satisfies these conditions. As the commutativity of (2.26), (2.27) with (4.3) is secured by construction, it is sufficient to solve (4.8) only by observing the subsidiary conditions for  $|\Phi_\varrho\rangle$ . Therefore the entire problem of solving (4.3) for bound states is reduced to the solution of (4.8), i.e. the solution of an equation for the function  $\varphi_\varrho(x_1 \dots x_\varrho)$ . Nevertheless the problem is not trivial. As in the kernel of the integralequation (4.8) the Green-functional

$$(m^2 - \Pi_\varrho \mathfrak{D}_h \mathfrak{D}^h \Pi_\varrho)^{-1}$$

appears, this Green-functional can be evaluated rigorously only if appropriate boundary conditions are imposed on the problem. In contrary to scattering theory, nothing is known about these conditions. In analogy to nonrelativistic quantum theory, we

guess that the condition has to be

$$(\mathfrak{T}(j, a) | \mathfrak{T}(j, a)) =: || \mathfrak{T}(j, a) ||$$

finite for all bound state solutions. To obtain a first impression, we assume for the Green-functional a Neumann series expansion

$$(m^2 - \Pi_\varrho \mathfrak{D}_h \mathfrak{D}_h \Pi_\varrho)^{-1} = \frac{1}{m^2} \sum_{l=0}^{\infty} \left( \frac{1}{m^2} \Pi_\varrho \mathfrak{D}_h \mathfrak{D}_h \Pi_\varrho \right)^l \quad (4.9)$$

Then for practical calculations it is necessary to truncate this infinite series for a finite value of  $l$ , e.g.  $l = N$ . Substituting this in (4.8), it becomes a tractable relativistic integral equation, whereas from  $|\Phi_r\rangle$  follows that in this approximation all  $q_n$  disappear for  $n > N + h$  with  $h$  fixed. Therefore by the truncation procedure Heisenbergs N.T.D.-procedure is reproduced in principle, although of course the N.T.D. equations are not exactly the same as those resulting from (4.8) by truncating with  $l = N$ . Further, it may be that the expansion (4.9) gives a hint for the choice of the appropriate boundary conditions. But also this problem has not been investigated so far. The solution procedure given above holds only for  $m \neq 0$ . As any particle has to be a bound state (or at least a resonance state) of the fundamental nonlinear spinor field, also the mass-zero-particle states have to appear among the manifold of possible eigenfunctionals of (2.20) resp. (2.23) and (2.17), (2.18). Therefore for these states the solution procedure has to be modified. This can be achieved by returning to the equation (2.20). Applying to (2.20) the causal Greenfunction  $D_\alpha^\beta(x)^{-1}$  for mass zero states an inhomogeneous term  $\partial_\beta(x) | \mathfrak{T}_0(j) \rangle$  has to be added, with

$$D_\alpha^\beta(x) \partial_\beta(x) | \mathfrak{T}_0(j) \rangle = 0. \quad (2.21 a)$$

Therefore in this case (2.20) goes over into

$$\begin{aligned} & [\partial_\kappa(x) + G_\kappa^\alpha(x - x') \{ V_{\alpha\beta\gamma\delta} [\partial_\beta(x') \partial_\gamma(x') \partial_\delta(x') \\ & + 3 F_{\beta\gamma}(0) \partial_\delta(x')] - i \varrho_0 j_\kappa(x) \} ] | \mathfrak{T}(j) \rangle \\ & = \partial_\kappa(x) | \mathfrak{T}_0(j) \rangle. \end{aligned} \quad (2.21 b)$$

Introducing the normal transform (2.22) and applying to the resulting equation  $j^\kappa(x) P_h(x)$  leads to

$$[\mathfrak{F}_h + \mathfrak{D}_h(j, d)] | \Phi(j) \rangle = \mathfrak{F}_h | \Phi_0(j) \rangle. \quad (4.1 a)$$

As the mass zero states have definite spin components the conditions (2.27) can be used for preparing (4.1 a). Applying  $\varepsilon^{3hjl} \mathfrak{M}_{jl}$  to (4.1 a) gives

together with (2.37)

$$[1 + \mathfrak{D}'(j, d)] | \Phi(j) \rangle = | \Phi_0(j) \rangle \quad (4.3 a)$$

with

$$\mathfrak{D}'(j, d) := \varepsilon^{3hjl} \mathfrak{M}_{jl} \mathfrak{D}_h(j, d) s_3^{-1}. \quad (4.3 b)$$

Assuming a Neumann series expansion of  $[1 + \mathfrak{D}']^{-1}$  to be valid, the solution of (4.3 a) is given by

$$| \Phi(j) \rangle = \sum_{l=0}^{\infty} \mathfrak{D}'(j, d)^l | \Phi_0(j) \rangle. \quad (4.3 c)$$

As the operators of (2.26), (2.27) commute with  $\mathfrak{D}'(j, d)$  the quantum numbers of  $| \Phi(j) \rangle$  are defined by those of  $| \Phi_0(j) \rangle$  or equivalently of  $| \mathfrak{T}_0(j) \rangle$ . Therefore  $| \mathfrak{T}_0(j) \rangle$  has to be a solution of (2.21 a) as well as of (2.17), (2.18). It can be verified easily that these conditions determine  $| \mathfrak{T}_0(j) \rangle$  completely and lead to the so-called local boson and fermion states given by  $\langle 0 | T \Psi_{\alpha_1}(x) \dots \Psi_{\alpha_n}(x) | a \rangle$ .

#### b) Scattering state calculations

From<sup>3, 26</sup> follows: the initial resp. final states of scattering states are free many particle states of dressed particles, i.e. of cluster-products applied to the groundstate. Therefore the boundary conditions of scattering states are given by cluster states. To construct them in functional space  $\mathfrak{F}$ , we proceed in analogy to the construction in  $\mathcal{H}$ . We consider a one particle bound state

$$\begin{aligned} | \Phi(j, \mathfrak{R}) \rangle &:= \sum_n \frac{1}{n!} \varphi_n^{\alpha_1 \dots \alpha_n}(x_1 \dots x_n | \mathfrak{R}) \\ &\quad \times j_{\alpha_1}(x_1) \dots j_{\alpha_n}(x_n) | \varphi_0 \rangle \end{aligned} \quad (4.10)$$

where  $\mathfrak{R}$  means the set of quantum numbers defining this particle state completely. It is assumed that such states are explicitly known by calculations according to a). Defining now the functional creation operator

$$\mathfrak{U}^+(\mathfrak{R}) := \sum_n \frac{1}{n!} \varphi_n^{\alpha_1 \dots \alpha_n}(x_1 \dots x_n | \mathfrak{R}) j_{\alpha_1}(x_1) \dots j_{\alpha_n}(x_n) \quad (4.11)$$

this state may be constructed by applying  $\mathfrak{U}^+(\mathfrak{R})$  to  $| \varphi_0 \rangle$ . Therefore we obtain

$$| \Phi(j, \mathfrak{R}) \rangle = \mathfrak{U}^+(\mathfrak{R}) | \varphi_0 \rangle. \quad (4.12)$$

Due to the definition of such creation operators we are able now to construct many particle states of free i.e. noninteracting clusters. In analogy to the Hilbertspace procedures introduced by WILDERMUTH<sup>37</sup>, EKSTEIN<sup>36</sup>, and HAAG<sup>38</sup> we define the rela-

tivistic functional cluster states by

$$|\Phi(j, \mathfrak{R}_1 \dots \mathfrak{R}_n) := \mathcal{A}[\mathfrak{U}^+(\mathfrak{R}_1) \dots \mathfrak{U}^+(\mathfrak{R}_n)]|\varphi_0\rangle \quad (4.13)$$

where (4.13) contains  $n$  noninteracting clusters with the quantum numbers  $\mathfrak{R}_1 \dots \mathfrak{R}_n$  and  $\mathcal{A}$  means the complete antisymmetrization of the product. But there are essential differences between the states (4.12) and (4.13). While the one particle cluster states (4.12) are solutions of (2.26), (2.27) and of (4.3), for the many particle states holds the following

*Stat. 4.1:* The many particle states (4.13) are solutions of (2.26) and (2.27) but in general not of (4.3).

*Proof:* Is given in <sup>26</sup>.

This result is also in analogy to the statements about nonrelativistic many particle cluster states by Wildermuth and Ekstein. The physical reason for this property of many particle states is obvious: As in nonlinear field theories the interaction cannot be switched out, only stable one particle solutions may exist. Any many particle solution has to be, therefore, a new stable solution, i.e. a new dressed one particle state or a scattering state. For the formation of a new stable particle, the quantum numbers of its cluster constituents may not be fixed. Therefore, if (4.13) would be a solution of (4.3), it should be a scattering state. But also scattering states cannot exist in general with fixed quantum numbers of its constituents. Therefore (4.13) cannot be a solution of (4.3). As the operators (4.11) create clusters, the following holds

*Stat. 4.2:* The algebra generated by the cluster creation operators  $\mathfrak{U}^+(\mathfrak{R})$  and their Hermitean conjugates  $\mathfrak{U}(\mathfrak{R})$  is not isomorphic to the algebra of the free particle operators  $\mathfrak{U}_f^+(\mathfrak{R})$  and  $\mathfrak{U}_f^-(\mathfrak{R})$

*Proof:* Is given in <sup>26</sup>.

Also this property can be verified for nonrelativistic clusters. There it is shown that cluster states in general are nonorthogonal<sup>36, 37</sup> Sec. III. But this results directly from statement 4.2 given here.

Therefore the dressed particle algebra is much more complicated than the free particle algebra. But this is no serious difficulty, as the complete cluster algebra is not required for practical calculations. Concerning the connection of the states (4.13) in  $\mathfrak{H}$  with the corresponding states in  $\mathcal{H}$ , the following statement can be given.

*Stat. 4.3:* The cluster states (4.13) in  $\mathfrak{H}$  can be mapped into the corresponding cluster states

$$|\mathfrak{R}_1 \dots \mathfrak{R}_n\rangle \text{ in } \mathcal{H}.$$

*Proof:* In general, the cluster states (4.13) belong to a reducible representation. Any reducible representation is characterized uniquely by its decomposition into irreducible parts. Therefore one may write

$$|\Phi(j, \mathfrak{R}_1 \dots \mathfrak{R}_n)\rangle = \sum_{\alpha} C_{\alpha}(\mathfrak{R}_1 \dots \mathfrak{R}_n) |\Phi(j, \alpha)\rangle^{\text{irr}} \quad (4.14)$$

where the expansion coefficients  $C_{\alpha}(\mathfrak{R}_1 \dots \mathfrak{R}_n)$  depend only on the quantum numbers  $\mathfrak{R}_1 \dots \mathfrak{R}_n$ , as no other characteristics are present. Therefore the reducible representation created by (4.13) can be characterized only by  $\mathfrak{R}_1 \dots \mathfrak{R}_n$ . As this characterization does not depend on the special representation space, it has to be true for states  $|\mathfrak{R}_1 \dots \mathfrak{R}_n\rangle$  in  $\mathcal{H}$  also. Then these states allow the expansion

$$|\mathfrak{R}_1 \dots \mathfrak{R}_n\rangle = \sum_{\alpha} \tilde{C}_{\alpha}(\mathfrak{R}_1 \dots \mathfrak{R}_n) |\alpha\rangle^{\text{irr}}. \quad (4.15)$$

Due to Stat. 3.3 a unitary mapping is established for irreducible representations between  $\mathcal{H}$  and  $\mathfrak{H}$ . As (4.14), (4.15) are a linear combination of irreducible representations, this statement holds also for (4.14), (4.15), q.e.d.

To apply the states (4.13) to the construction of scattering states, it is necessary to consider wave packets of (4.13). These are given by linear combinations

$$|\Phi(j, \kappa)\rangle := C_{\kappa}(\mathfrak{R}_1 \dots \mathfrak{R}_n) |\Phi(j, \mathfrak{R}_1 \dots \mathfrak{R}_n)\rangle \quad (4.16)$$

where  $\kappa$  denotes the quantum numbers of the packet. By a suitable choice of wave packets the configurations of the different clusters can be arranged to have an almost vanishing interaction. This is the starting point of  $S$ -matrix construction in ordinary Hilbert space. For the scattering functionals then the following statement can be derived

*Stat. 4.4:* The scattering functionals  $|\Phi^{(\pm)}(j, \kappa)\rangle$  for an initial or final configuration (4.16) decompose into

$$|\Phi^{(\pm)}(j, \kappa)\rangle = |\Phi(j, \kappa)\rangle + |\chi^{(\pm)}(j, \kappa)\rangle \quad (4.17)$$

where  $|\chi\rangle$  denotes the pure scattering part of  $|\Phi^{(\pm)}\rangle$ .

*Proof:* Assuming the scattering states to exist in  $\mathcal{H}$ , the relation (30) of<sup>36</sup> is valid in  $\mathcal{H}$ . Identifying



$|n\rangle$  with  $|\kappa\rangle$ , substitution of (30)<sup>36</sup> into (2.9) and application of statement 4.3 leads to the statement given here, q.e.d.

Finally we incorporate the boundary conditions into the dynamical calculations. Due to the translational invariance of any scattering process in Lorentz-space, the total mass of the system is conserved. As this total mass of the system is defined by the initial or final configuration of the type (4.16), it is given by the eigenvalue of  $\mathfrak{P}^2$  for clusters, derived in<sup>26</sup>. We denote this eigenvalue by  $m^2 = m_s^2$

(total mass of the system). Substitution of (4.17) into (4.3) for  $m^2 = m_s^2$  then gives the equation

$$(m^2 - \mathfrak{D}_h \mathfrak{D}^h) |\chi^{(\pm)}(j, \kappa)\rangle = -(m^2 - \mathfrak{D}_h \mathfrak{D}^h) |\Phi(j, \kappa)\rangle. \quad (4.18)$$

As  $|\Phi(j, \kappa)\rangle$  is assumed to be known, the Eq. (4.3) has been transformed, therefore, into an inhomogeneous equation. But for inhomogeneous equations no boundary conditions have to be settled, as the inhomogeneity itself determines the boundary values. Therefore Eq. (4.18) can be solved to give

$$|\chi^{(\pm)}(j, x)\rangle = -\lim_{\gamma \rightarrow 0} (m_s^2 - \mathfrak{D}_h \mathfrak{D}^h \pm i\gamma)^{-1} (m_s^2 - \mathfrak{D}_h \mathfrak{D}^h) |\Phi(j, x)\rangle. \quad (4.19)$$

Formally this equation corresponds completely Eksteins Eq. (32) of<sup>36</sup> by subtraction of  $|n\rangle$  in (32)<sup>36</sup> on both sides. But again the Green-functional has to be derived properly. Assuming for convenience a Neumann expansion like in (4.9), one obtains from (4.19) by applying the formulas of<sup>26</sup>

$$|\chi^{(\pm)}(j, x)\rangle = \frac{1}{m_s^2} \sum_{l=0}^{\infty} \left( \frac{1}{m_s^2} \mathfrak{D}_h \mathfrak{D}^h \right) C_{\kappa}(\mathfrak{R}_1 \dots \mathfrak{R}_n) \mathfrak{U} \left( \sum_{l=1}^n \mathfrak{U}^+(\mathfrak{R}_1) \dots \mathfrak{U}^+(\mathfrak{R}_{l-1}) \right. \\ \left. \cdot [(m_s^2 - \mathfrak{D}_h \mathfrak{D}^h) \mathfrak{U}^+(\mathfrak{R}_l)]_- \mathfrak{U}^+(\mathfrak{R}_{l+1}) \dots \mathfrak{U}^+(\mathfrak{R}_n) \right] |\varphi_0\rangle. \quad (4.20)$$

This solution corresponds to the common Born series. It may be used if no resonance scattering occurs. In the case of resonance phenomena the reciprocal has to be studied more thoroughly.

## 5. Functional Indefinite Metric

No comment has been given so far about the indefinite metric occurring in nonlinear spinor theory. It is caused by the noncanonical regularization procedure of the two point function  $F$  in the normal transform (2.22) if one observes the probabilistic interpretation of its spectral decomposition. Therefore the total state space of nonlinear spinor theory has to contain states with positive, zero and negative norm. It is not excluded a priori to give also states with negative norm a physical meaning. But in order to obtain a proper probabilistic interpretation of the theory for the theoretical description of physical processes, the states with positive norm and with negative norm have to be separable. This has been verified already for simple models<sup>39, 45</sup>. So one may assume that also in nonlinear spinor theory for physical processes the states of negative norm (ghost states) will not appear. But as this is a dynamical problem, it has not been clarified so far. So, concerning the role of ghoststates in nonlinear spinor theory, one depends on assumptions which have to be justified later by the result of calculational experience. The most simple version seems to

be the following: We denote the total state space of nonlinear spinor theory by  $\mathcal{V}$  and identify the subspace of physical states with the space  $\mathcal{H}$  defined in (1.1), whereas the subspace of ghosts and of states with vanishing norm is denoted by  $\mathcal{H}^g$ . Then  $\mathcal{V} = \mathcal{H} \cup \mathcal{H}^g$ . Assuming now that  $\mathcal{H}$  and  $\mathcal{H}^g$  are separated completely with respect to spinor dynamics, i.e. scattering processes, one may consider only the map of  $\mathcal{H}$  into an appropriate functional state space  $\mathfrak{H}$ , whereas  $\mathcal{H}^g$  is omitted, as it does not contribute to physics. Then the functional state space  $\mathfrak{H}$  is a positive definite state space and its metrical fundamental tensor with respect to its base vectors has to be a positive definite quantity. This point of view is assumed tacitly in the preceding sections, as the base functionals (2.6) lead to the positive definite scalar product (2.8). Therefore in the functional version the ghoststates are excluded ab initio and functionally one does not know anything about them. This procedure may be justified, if one knows definitely that the separation of physical states and of ghoststates is perfect. On the other hand, if this separation is doubtful and has to be investigated, it may be more advantageous to consider the functional map  $\mathfrak{B}$  of the total state space

$\mathcal{V}$  of nonlinear spinor theory. As  $\mathcal{V}$  is an indefinite space,  $\mathfrak{B}$  has it to be also, i.e. the metrical fundamental tensor of its base states has to be an indefinite quantity, too. In this case a more general construction of a functional space has to be performed, than that given in<sup>18</sup>. This has been mentioned already in Section 2. Performing this construction it will throw a light also on the role of ghoststates in functional space. To show this, we refer to the expansion of  $j_\alpha(x)$  given in<sup>18</sup>. It reads

$$j_\alpha(x) = \sum_{\kappa} \int C_\alpha(p|\kappa) e^{ipx} a_\kappa^+(p) d^4p \quad (5.1)$$

where the integration runs over the entire Lorentz space. (5.1) can be divided into invariant subspace integrations

$$j_\alpha(x) = \sum_{\kappa} \left[ \int_{L_t} C_\alpha^t(p|\kappa) e^{ipx} a_\kappa^{t+}(p) d^4p + \int_{L_s} C_\alpha^s(p|\kappa) e^{ipx} a_\kappa^{s+}(p) d^4p \right] \quad (5.2)$$

where  $L_t$  denotes the manifold of timelike or lightlike vectors  $p^2 \geq 0$  and  $L_s$  the spacelike vectors with  $p^2 < 0$ . To define the quantities occurring in (5.2), we observe that a division into classes is possible by the invariant mass values  $p^2 = m^2$  (timelike),  $p^2 = \mu^2$  (spacelike). Then a class is defined by a fixed value of  $m$  or  $\mu$  and each  $p$  can be contained only in one class. Denoting these classes by  $C_m$  and  $C_\mu$ , we may put

$$\begin{aligned} a_\kappa^{t+}(p) &:= a_\kappa^+(p, m), & p \in C_m, \\ a_\kappa^{s+}(p) &:= a_\kappa^+(p, \mu), & p \in C_\mu \end{aligned} \quad (5.3)$$

where  $a^+(p, m)$  and  $a^+(p, \mu)$  are creation operators of ordinary free relativistic fields of mass  $m$  or mass  $i\mu$ . It has been shown by direct calculation<sup>18, 21</sup>, that also the fields with imaginary masses show the proper transformation law of Dirac spinors and are thus admitted for the spectral decomposition (5.1). Now the functional state space is generated by the power functionals (2.6) and their scalar product leads to the metrical fundamental tensor of this space given by (2.8). As is wellknown, this scalar-product depends on the operator properties of  $j_\alpha(x)$  expressed by the anticommutators of  $j_\alpha(x)$  and  $j_\alpha(x)^+$ . By the spectral decomposition (5.1) these anticommutators depend themselves on those of the creation operators  $a^+(p)$  and  $a(p)$ . For (2.1) they are assumed to be

$$[a_\kappa(p, M) a_\kappa^+(p', M')]_+ = \delta_{MM'} \delta_{\kappa\kappa'} \delta(p - p') \quad (5.4)$$

where  $M$  denotes  $m$  as well as  $\mu$  masses, and all other anticommutators vanish. In consequence of (5.4) one obtains then a state space with positive definite metric. Now it is obvious that (5.4) are not the most general anticommutators which can be found. Assuming the anticommutators

$$[a_\kappa(p, M) a_\kappa^+(p', M')]_+ = g(M) \delta_{MM'} \delta_{\kappa\kappa'} \delta(p - p') \quad (5.5)$$

by explicit construction, a representation can be given which leads also to Poincare invariant functional operators. But in general the anticommutators of  $j_\alpha(x)$  and  $j_\alpha^+(x)$  are different from (2.1), i.e. the identification (2.4) is not allowed. Especially if  $g(M)$  is defined in  $-\infty < g(M) < \infty$ , one obtains with (2.7) a scalar product

$$\langle D_n | D_m \rangle = g_{nm} \quad (5.6)$$

being an indefinite quantity. Therefore by (5.5) functional spaces can be generated which are representation spaces of the Poincare-group with indefinite metric. By an appropriate choice of  $g(M)$ , therefore, ghoststates may be introduced in the functional theory. Using these state spaces, one has to distinguish between co- and contravariant quantities. These are defined by the introduction of a second set of base vectors  $\langle D^n |$  satisfying the conditions

$$\langle D^n | D_m \rangle = \delta_m^n. \quad (5.7)$$

From (2.1) follows that  $\langle D^n |$  has to be represented by

$$\langle D^n(x_1 \dots x_n) | := \langle q_0 | \partial_{\alpha_1}(x_1) \dots \partial_{\alpha_n}(x_n) \quad (5.8)$$

$$0 \leq n \leq \infty$$

and by common procedures one obtains

$$\langle D_m | = \sum_j g_{mj} \langle D^j |. \quad (5.9)$$

Then by (5.9) the connection between  $\partial_\alpha(x)$  and  $j_\alpha(x)^+$  can be established. Therefore all operators in functional space required for the functional quantum theory are well defined by construction. As the corresponding calculations are rather extensive, we do not go into detail here. But we mention only that by (3.2) and (3.3) the functional scalar product contains now the quantity  $g_{nm}$  showing explicitly the dependence of this product on the indefinite metric. Especially the operator  $\mathfrak{B}$  of (3.4) has to be generalized to

$$\mathfrak{B} = \sum | D^n \rangle g_{nm} \langle D^m | \varepsilon^{n/2} \varepsilon^{m/2} \quad (5.10)$$

There seems to be no other natural way to introduce the indefinite metric in functional space. But the decision to use the definite state space for a functional map into  $\mathcal{H}$  or the indefinite state space for a functional map into  $\mathcal{V}$  requires calculational experiences for a comparison of both methods.

## 6. Conclusions

We assume that nonlinear spinor theory exists, i.e. can be realized in a representation space  $\mathcal{H}$ . Then, by functional quantum theory an isomorphic map with that theory can be established in a functional space  $\mathfrak{F}$ . Nevertheless, the assumption of a representation in  $\mathcal{H}$  is of academic interest only. So far no calculations concerning the observables of this theory can be done in  $\mathcal{H}$ .

On the other hand, by functional quantum theory the entire framework for the calculation of all global observables has been developed in the preceding discussions. *Therefore in the functional version of the theory one knows more about it than in the conventional version.* Keeping this in mind, one may argue in the following way: Functional quantum theory is the only really tractable version of nonlinear spinor

theory, where all so far unsolved problems concerning the definition and calculation of global observables can now be solved. Therefore it is justified to forget the untractable conventional version and to formulate nonlinear spinor field quantum theory in functional space only i.e. *to establish nonlinear spinor field quantum theory a priori in functional space.* As can be seen from the foregoing considerations, there is still much work to be done concerning the existence of the theory in a rigorous mathematical framework. On the other hand it is wellknown that a mathematical progress is possible only if the problems are formulated properly from the physical point of view. It is my opinion that in this direction the foundation of nonlinear spinor theory in the framework of functional quantum theory and relativistic functional cluster representations is a considerable progress.

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